Performance Analysis of Multiuser Detection and Decoding for Subcarrier Hopping Multiple Access

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Abstract—Real-time wireless communications such as motion controls of vehicles and machines have been attractive in recent years, in which latency and reliability are of significant importance. A new orthogonal frequency-division multiplexing (OFDM)-based uplink multiple access system called subcarrier hopping multiple access (SHMA) can achieve high reliability due to both coding gain and frequency diversity gain without increasing decoding delay by combining super-orthogonal convolutional codes (SOCC) with subcarrier hopping. In order to improve the spectral efficiency, multiple users should share the same set of subcarriers together with multiuser detection and decoding (MUD) at the receiver to mitigate multiple access interference. In this work, we focus on two representative MUD schemes, i.e., super-trellis decoding (STD) and successive interference cancellation (SIC), and develop approximate bit error rate expressions when each MUD scheme is employed. Through the comparison with simulation results, we demonstrate that our analysis is useful for predicting the behavior of MUD. Also shown is the effectiveness of our proposed hopping design in fully achieving frequency diversity gain over static fading channels.

I. INTRODUCTION

Future wireless communications should support various requirements imposed by diverse applications. Among them, motion controls of machines including unmanned aerial vehicles (UAV) typically require higher reliability with even lower latency. This work thus focuses on the realization of low latency wireless systems without sacrificing reliability.

In order to achieve this goal, the authors have recently proposed a new uplink multiple access system called subcarrier hopping multiple access (SHMA) [1]. It is based on orthogonal frequency-division multiplexing (OFDM) with super-orthogonal convolutional code (SOCC) [2]–[4]. The SOCC is a class of very low-rate convolutional codes that have powerful error correction capability even with simple decoding structure based on Viterbi algorithm. It can thus achieve good error rate performance even in low SNR without resorting to iterative decoding that leads to increasing latency. Since it is based on the conventional OFDM, the inter-symbol interference (ISI) caused by the frequency selectiveness of wireless channels can be avoided by the insertion of cyclic prefix (CP) without complicated equalization at the receiver.

While OFDM can achieve high spectral efficiency as well as robustness against frequency selective fading channels, it suffers from high peak-to-average power ratio (PAPR), which results in low power efficiency at a power amplifier (PA) [5], [6]. The proposed SHMA system can solve this PAPR problem by employing Golay sequence [7] as the decoder outputs such that the resulting signal PAPR is as low as 3 dB [8]. This power efficiency improvement enhances the transmission range. Moreover, by incorporating subcarrier hopping, the frequency diversity gain can be fully exploited.

The proposed SHMA system allows multiple users to share the identical subcarriers such that the overall spectral efficiency can be enhanced. This subcarrier assignment, however, does not retain the orthogonality among users and thus leads to multiple access interference (MAI). Nevertheless, it can be suppressed by multiuser detection and decoding (MUD) at the receiver. In our previous work, the diversity order achieved by the introduction of the subcarrier hopping is analyzed in [1]. Comparison of error rate performance and spectral efficiency with orthogonal frequency-division multiple access (OFDMA) is also examined in [1]. Furthermore, based on the two commonly adopted MUD schemes, i.e., successive interference cancellation (SIC) and super-trellis decoding (STD), the trade-off between performance and complexity is discussed in [9] but theoretical analysis was left as future work.

While several SIC-based algorithms for OFDM systems are proposed [10], performance analysis for SIC in the framework of coded OFDM has not been well studied in the literature. Therefore, in this paper, we attempt to develop approximate lower bound expressions of bit error rate (BER) for the proposed SHMA system employing STD and SIC. We also propose a hopping pattern design that can fully exploit the available diversity, and its effectiveness is evaluated through the comparison between the developed theoretical bound and simulated BER.

II. SHMA SYSTEM

We consider an uplink of multiuser communications in which multiple users transmit their signals to a single base station in a quasi-synchronous manner such that the interference among users on different channels (i.e., inter-carrier interference among all the users) is negligible. We assume that each of the transmitter and receiver has only a single antenna. Furthermore, perfect frequency synchronization is assumed such that carrier frequency offset (CFO) is negligible.
at the receiver. We note that the CFO is a crucial problem for the uplink of multiuser OFDM systems [11], and thus development of estimation and compensation of CFO suitable for real-time communications is important, which is left as our future work. The key techniques that realize the proposed system are the SOCC combined with Golay sequence and subcarrier hopping with OFDM, which we review in what follows.

A. SOCC with Golay Sequence

The SOCC is a class of very low-rate convolutional codes. By assigning an orthogonal sequence as an output of the convolutional encoder, SOCC efficiently improves its distance spectrum as the constraint length \( K \) increases and the code rate becomes lower. In fact, the minimum free distance \( d_t \) of the SOCC with the constraint length \( K \) is given by [12]

\[
d_t = 2^{K-3}(K + 2).
\]  

(1)

Figure 1 shows an encoder structure of SOCC. In the encoding process, the middle \( K - 2 \) bits are used for selection of an orthogonal sequence of length \( 2^{K-2} \), and then the polarity of the selected sequence is determined by an XOR operation with the two outer bits. Consequently, the code rate is denoted by \( R_c = 1/2^{K-2} \).

In the original SOCC, a Walsh-Hadamard (WH) sequence is used as an orthogonal sequence. On the other hand, a set of orthogonal Golay sequences [13] is employed in the proposed SHMA system instead of that of WH sequences in order to reduce the PAPR of the resulting OFDM signals. The SOCC employing Golay sequences as its outputs has the same distance spectrum (and transfer function) as that with WH sequences [9], [13]. This fact implies that the application of Golay sequences to the SOCC does not affect the error correcting capability of the code.

The output sequence consisting of \( N_u = 2^{K-2} \) bits is mapped onto BPSK constellation with unit average power, and then allocated to equally spaced \( N_s \) subcarriers in an \( N_s \)-subcarrier OFDM symbol. Note that equal subcarrier spacing is required to guarantee that the resulting OFDM signal has a 3 dB PAPR. In this work, the subcarrier space is chosen as large as possible such that the achievable frequency diversity effect is maximized (i.e., minimizing the statistical correlation among modulated subcarriers). Consequently, the resulting subcarrier space is given by \( J = \lfloor N_s/N_u \rfloor \) where \( \lfloor x \rfloor \) is the maximum integer smaller than or equal to \( x \).

The SOCC can be decoded by the conventional soft-decision Viterbi decoder, which leads to the receiver implementation with low complexity. Furthermore, since it is suitable to parallel implementation of add-compare-select (ACS) circuits, the decoding latency can be made significantly lower than those involving iterative decoding process.

B. Subcarrier Hopping

In real-time communications, it is generally assumed that short-frame transmission is employed, in which the channel is assumed to be static during the transmission of a single frame consisting of \( M \) OFDM symbols (corresponding to a single codeword). If the same set of subcarriers is allocated to each user for all the OFDM symbols in a single frame as in the case of the conventional OFDMA or interleaved frequency-division multiple access (IFDMA) [14], it fails to achieve the full frequency diversity gain offered by the channel. Therefore, our system employs subcarrier hopping approach [1], which is summarized as follows.

We consider the multiple access system where \( N_t \) active users share \( N_s \) subcarriers in each OFDM symbol with each user selecting its own set of \( N_s = 2^{K-2} \) subcarriers with the space \( J \). Let \( \mathcal{N}_t^{(m)} = \{ k_{1,1}^{(m)}, k_{1,2}^{(m)}, \ldots, k_{1,N_s}^{(m)} \} \) denote the set of subcarrier indices allocated to the \( i \)th user in the \( m \)th OFDM symbol, where \( m \in \{ 1, 2, \ldots, M \} \) and \( i \in \mathcal{U} \), with \( \mathcal{U} = \{ 1, 2, \ldots, N_t \} \) representing the set of the active user indices. Here, \( k_{i,n}^{(m)} \in \{ 0, 1, \ldots, N_s - 1 \} \) represents the subcarrier index onto which the BPSK symbol corresponding to the \( n \)th bit of the \( i \)th user’s SOCC encoder output (of length \( N_s \)) is mapped. For example, in the conventional OFDMA system, since users utilize the same set of subcarriers over all OFDM symbols, \( \mathcal{N}_t^{(m)} \) is invariant for any OFDM symbol index \( m \). On the other hand, in our proposed system employing the subcarrier hopping, \( \mathcal{N}_t^{(m)} \) varies by each OFDM symbol transmission. Specifically, for the first OFDM symbol (i.e., \( m = 1 \)), the initial subcarrier index \( k_{1,1}^{(1)} \) is chosen randomly from the set \( \{ 0, 1, \ldots, J - 1 \} \) and the remaining subcarrier indices are separated by the interval \( J \) such that the transmitted OFDM signals have 3 dB PAPR with maximizing the frequency diversity effect, i.e., \( k_{1,n}^{(1)} = k_{1,1}^{(1)} + (n - 1)J \) for \( n = 2, 3, \ldots, N_s \). For the other OFDM symbols (i.e., \( m = 2, \ldots, M \)), the initial subcarrier index is determined as \( k_{1,1}^{(m)} = (k_{1,1}^{(m-1)} + V) \mod J \) where

\[
V = \begin{cases} 
1, & \text{if } \lfloor J/K \rfloor = 0, \\
\lfloor J/K \rfloor, & \text{otherwise}
\end{cases}
\]  

(2)

denotes the hopping interval between the consecutive OFDM symbols. This hopping design guarantees that distinct sets of subcarriers are used over at least \( K \) consecutive OFDM symbols (i.e., \( K \) trellis segments). In this manner, the consecutive bits in the trellis do not suffer from the same amount of fading, thus contributing to the increase of diversity gain. The remaining subcarrier indices (\( k_{1,n}^{(m)} \) for \( n = 2, \ldots, N_s \)) are
separated by the interval \( J \) similar to the first OFDM symbol. A CP is added to each OFDM symbol, and we assume that the length of CP is long enough such that the effect of ISI associated with delay spread of the channel is negligible.

III. Multiuser Detection and Decoding

The \( k \)th subcarrier of the \( n \)th received OFDM symbol that is simultaneously shared by \( N_u \) users is represented by

\[
Y_k^{(m)} = \sum_{i \in S_k^{(m)}} H_{i,k} X_{i,k}^{(m)} + N_k^{(m)}. \tag{3}
\]

Here, \( X_{i,k}^{(m)} \in \{-1,1\} \) denotes the \( i \)th user’s transmitted BPSK symbol, \( N_k^{(m)} \) is an AWGN term which follows \( \mathcal{CN}(0,N_0) \) and \( S_k^{(m)} = \{i_1,i_2,\ldots\} \) denotes the set of the user indices allocated to the \( k \)th subcarrier of the \( m \)th OFDM symbol. Moreover, \( H_{i,k} \) denotes the channel coefficient of the \( k \)th subcarrier over the \( i \)th user’s channel. It follows that

\[
H_{i,k} = \sum_{\ell=1}^{L} \sqrt{p_{i,\ell}} h_{i,\ell} e^{-j2\pi f_{\ell} n_k}, \tag{4}
\]

where \( p_{i,\ell}, h_{i,\ell} \), and \( L \) denote the power delay profile (PDP), the channel impulse response, and the number of channel taps, respectively. Note that \( H_{i,k} \) is invariant for each OFDM symbol transmission since the static frequency-selective fading channel is assumed. The perfect channel state information (CSI) is assumed to be available at the receiver side.

In the case of no MAI (i.e., \( |S_k^{(m)}| = 1 \) for any pair of \( k \) and \( m \)), the received symbol \( Y_k^{(m)} \) can be directly used in metric calculation of the conventional Viterbi decoder designed for fading channels. Otherwise, MUD is necessary to mitigate or eliminate the MAI. In the following, we will introduce the two representative MUD schemes, i.e., super-trellis decoding (STD) and the decoding based on successive interference cancellation (SIC). The former achieves optimum performance at the cost of high complexity, whereas the latter generally achieves suboptimum performance but with much less complexity.

A. Super-Trellis Decoding

The STD is a class of maximum-likelihood sequence estimation (MLSE) for trellis-based codes such as convolutional codes and turbo codes [15] which jointly detects all users’ sequences. Specifically, in the STD, Viterbi algorithm is performed over a super-trellis which is a single trellis representation generated by merging all users’ trellises to choose the most likely sequence of all the users.

While the STD can achieve optimum error rate performance regardless of the number of users due to the MLSE-based decoding [16], its computational complexity is significantly high stemming from the large number of states of the super trellis, which generally increases exponentially with the number of users. Specifically, if \( N_u \) users have the same SOCC encoder with constraint length \( K \), the number of the super-trellis states results in \( 2^{N_u (K-1)} \).

B. Successive Interference Cancellation

In the proposed SIC-based decoding, we sequentially select the target user to be decoded in the order of the received signal power. At the first step, the user with the highest channel gain is selected as the target user. Specifically, the target user \( u \in U \) is determined by the following rule:

\[
u = \arg \max_{i \in U} \sum_{\ell=1}^{L} |h_{i,\ell}|^2. \tag{5}
\]

We then calculate the bit metrics and decode the signal of the target user. For this target user, the other users’ signals are regarded as interference. In our decoding algorithm, the sum of the interference and noise terms is assumed to be Gaussian, which is represented by

\[
N_k^{(m)} = \sum_{i \in S_k^{(m)} \setminus u} H_{i,k} X_{i,k}^{(m)} + N_k^{(m)}, \tag{6}
\]

where \( N_k^{(m)} \) follows \( \mathcal{CN}(0, I_k^{(m)} + N_0) \) and \( I_k^{(m)} \) is given by

\[
I_k^{(m)} = \sum_{i \in S_k^{(m)} \setminus u} |H_{i,k}|^2. \tag{7}
\]

Under this assumption, the metric of the bit carried by the \( k \)th subcarrier in the \( m \)th OFDM symbol of user \( u \), denoted by \( c_{u,k}^{(m)} \), is calculated as

\[
X_{c_{u,k}}^{(m)} = \log f(Y_k^{(m)} \mid c_{u,k}^{(m)} = b, H_{u,k}) \tag{8}
\]

\[
= \log \frac{1}{\pi I_k^{(m)} + N_0} \exp \left( -\frac{|Y_k^{(m)} - H_{u,k}(2b - 1)|^2}{I_k^{(m)} + N_0} \right), \tag{9}
\]

where \( f(\cdot) \) denotes the conditional probability density function (PDF). The calculated bit metric is fed into the Viterbi decoder, and then the estimated symbol of the target user, \( \hat{X}_{u,k}^{(m)} \), is obtained. The estimated symbol is subtracted from the received symbol (3) as follows:

\[
Y_k' = Y_k^{(m)} - H_{u,k} \hat{X}_{u,k}^{(m)}. \tag{10}
\]

The resulting symbol after interference cancellation is used for decoding of the next candidate user. Before proceeding to the next decoding iteration, \( U \) should be updated by excluding the index of the selected user.

IV. Performance Analysis

In this section, we develop an approximate BER lower bound for the SHMA system employing the STD and SIC. Firstly, we make a brief review of the standard union bound analysis for binary convolutional codes. The bit error probability \( P_b \) is bounded by [4]

\[
P_b \leq \frac{1}{K \cdot \sum_{d_H=d_t}^{\infty} \beta(d_H) P(d_H)} \tag{11}
\]
where \(d_{f}\) is the minimum free distance of the convolutional codes, \(\beta(d_{f})\) denotes the total input weight of the error events with the Hamming distance \(d_{f}\), and \(P(d_{f})\) denotes the codeword pairwise error probability (PEP) with the Hamming distance \(d_{f}\). Moreover, \(k_{c}\) is the number of the information bits per trellis segment, and it is equal to one throughout this work since the code rate of SOCC is \(R_{c} = 1/2K-2\).

In general, the union bound serves as a tight upper bound for high SNR, but its exact calculation requires prohibitively high computational complexity. In our analysis, we take into account only the error event with the minimum Hamming distance \(d_{f}\) for simplicity. This in turn corresponds to a lower bound, given by

\[
P_{b} \geq \frac{1}{k_{c}} \beta(d_{f}) P(d_{f}). \tag{12}\]

This bound becomes tight as SNR increases, since the error event with \(d_{f}\) is dominant in all the error events. For SOCC, \(\beta(d_{f})\) can be readily obtained from the transfer function of codes presented in [4]. In the following, we thus focus on calculating the codeword PEP \(P(d_{f})\) for the STD and SIC.

In order to make our analysis tractable, we consider a simple scenario where all \(N_{u}\) users share the identical set of subcarriers for all OFDM symbols. Specifically, we consider the case \(\mathcal{N}^{(m)}_{s} = \mathcal{N}^{(m)}_{a} = \cdots = \mathcal{N}^{(m)}_{N_{u}} = \mathcal{N}^{(m)}\), where \(\mathcal{N}^{(m)}\) corresponds to the set of subcarrier indices to which \(N_{u}\) users are allocated. We further assume that

\[
|\mathcal{S}_{k}^{(m)}| = \begin{cases} N_{u}, & k \in \mathcal{N}^{(m)}, \\ 0, & k \notin \mathcal{N}^{(m)}, \end{cases} \tag{13}\]

for any \(m \in \{1, 2, \cdots, M\}\), where \(|\mathcal{A}|\) represents the cardinality of an arbitrary set \(\mathcal{A}\). Note that this scenario can be considered as the worst case where the hopping patterns of all the users happen to be identical and thus MAI exists in all allocated subcarriers. Note that, in practice, the hopping patterns can be coordinated such that some of the allocated subcarriers are MAI free. Therefore, the theoretical BER to be derived in this section can be lower bound for such practical scenarios. For simplicity, we further assume that all the channels are modeled by the Rayleigh fading with equal-power \(L\)-taps. Therefore, \(h_{i,\ell}\) follows \(\mathcal{CN}(0, 1)\) and \(p_{i,1} = \cdots = p_{i,L} = 1/L\) for \(i = 1, \cdots, N_{u}\).

\section*{A. PEP Analysis for STD}

Since the STD is MLSE-based decoding, it can perfectly eliminate the interference as long as each user experiences statistically independent realization of channel, and its achievable BER is identical to the single-user case without MAI. Therefore, the PEP is given by [17]

\[
P_{\text{STD}}(d_{f} \mid H_{i}) = Q\left(\sqrt{\frac{\sum_{k \in \mathcal{W}} d_{\min}^{2} |H_{i,k}|^{2}}{2N_{0}}} \right), \tag{14}\]

where \(H_{i} = \{H_{i,1}, \cdots, H_{i,N_{u}}\}\) denotes the vector of channel coefficients for the \(i\)th user, \(d_{\min}\) is the minimum Euclidean distance of the constellation, and \(\mathcal{W} = \{w_{1}, \cdots, w_{L}\}\) denotes the subset of the subcarrier indices that correspond to the bit error event.

For simplicity of analysis, we assume the full diversity condition where the sum of channel coefficients is approximate by

\[
\sum_{k \in \mathcal{W}} |H_{i,k}|^{2} \approx \begin{cases} \frac{d_{f}}{L} \sum_{\ell=1}^{L} |h_{i,\ell}|^{2}, & \text{if } L \leq d_{f}, \\
\sum_{\ell=1}^{d_{f}} |h_{i,\ell}|^{2}, & \text{if } L > d_{f}. \end{cases} \tag{15}\]

In the following, we will focus on the case of \(L \leq d_{f}\) since the minimum free distance of SOCC in (1) can be much larger than a typical diversity order in our system model.

Using the above approximation, the PEP of (14) is rewritten as

\[
P_{\text{STD}}(d_{f} \mid H_{i}) \approx Q\left(\sqrt{\frac{d_{\min}^{2} \frac{d_{f}}{L} \sum_{\ell=1}^{L} |h_{i,\ell}|^{2}}{2N_{0}}} \right) \tag{16} = \frac{1}{\pi} \int_{0}^{\pi/2} \exp \left( -\frac{d_{\min}^{2} \frac{d_{f}}{L} \sum_{\ell=1}^{L} |h_{i,\ell}|^{2}}{4N_{0} \sin^{2} \theta} \right) \sin \theta d\theta \tag{17} = \frac{1}{\pi} \int_{0}^{\pi/2} \prod_{\ell=1}^{L} \exp \left( -\frac{d_{\min}^{2} d_{f} |h_{i,\ell}|^{2}}{4N_{0} \sin^{2} \theta} \right) d\theta. \tag{18}\]

By the assumption of Rayleigh fading, \(|h_{i,\ell}|^{2}\) follows exponential distribution with its PDF given by \(f_{b}(x) = e^{-x}\). By taking expectation with respect to \(|h_{i,\ell}|^{2}\), the unconditional PEP is expressed as

\[
P_{\text{STD}}(d_{f}) \approx \frac{1}{\pi} \int_{0}^{\pi/2} \left( \frac{1}{1 + \frac{d_{\min}^{2} d_{f}}{4N_{0} \sin^{2} \theta}} \right)^{L} d\theta. \tag{19}\]

\section*{B. PEP Analysis for SIC}

For simplicity of analysis, we consider the two-user case with the users denoted by \(U_{1}\) and \(U_{2}\), where \(U_{1}\) experiences higher channel gain and thus is to be decoded first, followed by decoding of \(U_{2}\). It should be noted that extension of our analysis to the case of more than two users is straightforward. In the two-user case, the \(k\)th subcarrier of the \(n\)th received OFDM symbol is represented by

\[
Y_{k}^{(m)} = H_{1,k} X_{1,k}^{(m)} + H_{2,k} X_{2,k}^{(m)} + N_{k}^{(m)}. \tag{20}\]

In the following, we derive a PEP for each user.

\begin{enumerate}
\item \textbf{PEP Expression for }\(U_{1}\): For decoding of \(U_{1}\), the \(H_{2,k} X_{2,k}^{(m)}\) is considered as interference, and we assume that this interference term follows Gaussian. Therefore, (20) is rewritten as

\[
Y_{k}^{(m)} = H_{1,k} X_{1,k}^{(m)} + N_{1,k}^{(m)}\tag{21}\]

where

\[
N_{1,k}^{(m)} = H_{2,k} X_{2,k}^{(m)} + N_{k}^{(m)}, \tag{22}\]

which follows \(\mathcal{CN}(0, \sigma_{1}^{2})\) with

\[
\sigma_{1}^{2} = |H_{2,k}|^{2} + N_{0}. \tag{23}\]
\end{enumerate}
Consequently, the conditional PEP for $U_1$ is expressed as

$$P_1(d_t \mid H_1, H_2) = Q\left(\sqrt{\frac{d_{\text{min}}^2 \sum_{k \in W_0} |H_{1,k}|^2}{2\sigma_t^2}}\right)$$

(24)

$$= Q\left(\sqrt{\frac{d_{\text{min}}^2 \sum_{k \in W_0} |H_{1,k}|^2}{2|H_{2,k}|^2 + 2N_0}}\right).$$

(25)

From (4), the following approximation can be derived:

$$|H_{2,k}|^2 \approx \frac{1}{L} \sum_{\ell=1}^{L} |h_{2,\ell}|^2.$$  

(26)

Using the above approximation and (15), (25) is approximated as

$$P_1(d_t \mid H_1, H_2) \approx Q\left(\sqrt{\frac{d_{\text{min}}^2 d_t |\alpha_1|}{2\sigma_t^2 + 2N_0}}\right),$$

(27)

where $\alpha_i = \sum_{\ell=1}^{L} |h_{i,\ell}|^2$. In the SIC decoding, the decoding priority is determined based on (5). Therefore, $h_{1,\ell}$ and $h_{2,\ell}$ are not independent and they have the relationship of $\alpha_1 > \alpha_2$. Note that the sum of independent and identically distributed (i.i.d.) random variables each with exponential distribution $f_\alpha(x) = e^{-x}$ follows the chi-square distribution with $2L$ degrees of freedom whose PDF is expressed as

$$f_\alpha(x) = \frac{e^{-x(L-1)}}{(L-1)!}.$$  

(28)

Using the above approximation and (15), (25) is approximated as

$$P_1(d_t \mid H_1, H_2) \approx Q\left(\sqrt{\frac{d_{\text{min}}^2 d_t \alpha_1}{2\sigma_t^2 + 2N_0}}\right),$$

(27)

where $\alpha_i = \sum_{\ell=1}^{L} |h_{i,\ell}|^2$. In the SIC decoding, the decoding priority is determined based on (5). Therefore, $h_{1,\ell}$ and $h_{2,\ell}$ are not independent and they have the relationship of $\alpha_1 > \alpha_2$. Note that the sum of independent and identically distributed (i.i.d.) random variables each with exponential distribution $f_\alpha(x) = e^{-x}$ follows the chi-square distribution with $2L$ degrees of freedom whose PDF is expressed as

$$f_\alpha(x) = \frac{e^{-x(L-1)}}{(L-1)!}.$$  

(28)

Based on the order statistics, a joint PDF of $\alpha_1$ and $\alpha_2$ under the condition of $\alpha_1 > \alpha_2$ is given by

$$f_{\alpha_1,\alpha_2}(x_1, x_2) = 2! f_\alpha(x_1) f_\alpha(x_2),$$

(29)

and thus an approximate form of the unconditional PEP is obtained as

$$P_1(d_t) \approx$$

$$\int \int_{D} Q\left(\sqrt{\frac{d_{\text{min}}^2 d_{\text{max}} x_1}{2\sigma_t^2 + 2N_0}}\right) f_{\alpha_1,\alpha_2}(x_1, x_2) dx_1 dx_2,$$

(30)

where $D = \{(x_1, x_2) \mid 0 < x_2 < x_1 < \infty\}$ denotes the domain of integration.

2) PEP Expression for $U_2$: On the bit metric calculation of the SIC for decoding of $U_2$, the reference symbol depends on whether the decoding of $U_1$ is successful or not. Since exact analysis involves a prohibitive number of probabilistic events, we take into account only the two tractable cases in what follows. The first case is that decoding of $U_1$ is successful. The second case is that the error event of $U_1$ with $d_t$ and that of $U_2$ occur in a synchronous manner, starting at the same trellis segment. For the former case, the symbol of $U_1$ (which is interference to $U_2$) is successfully removed from the interfered symbol at the cancellation process by (10). On the other hand, for the latter case, the interference term is augmented and the reference symbol after the process of (10) is given by

$$Y_k' = H_{2,k}X_{2,k} + 2H_{1,k}X_{1,k} + N_k^{(m)}.$$  

(31)

By assumption, $N_{2,k}^{(m)}$ follows $CN(0, \sigma_2^2)$ with its variance given by

$$\sigma_2^2 = 4|H_{1,k}|^2 + N_0.$$  

(32)

Combining the above two cases, the approximate conditional PEP of $U_2$ is given by

$$P_2(d_t \mid H_1, H_2)$$

$$\approx (1 - P_1(d_t \mid H_1, H_2)) \int Q\left(\sqrt{\frac{d_{\text{min}}^2 d_{\text{max}} x_1}{2\sigma_t^2 + 2N_0}}\right) f_{\alpha_1,\alpha_2}(x_1, x_2) dx_1 dx_2,$$

(30)

where $P_1(d_t \mid H_1, H_2)$ is given by (27). Finally, similar to (30), the unconditional PEP is derived by integrating (33) with the joint PDF of $\alpha_1$ and $\alpha_2$ given by (29).

V. NUMERICAL RESULTS

In the following simulation, the SOCC with constraint length $K = 4$ is employed where $d_t = 12$. The number of OFDM symbols $M$ and that of subcarriers $N_s$ are chosen as 768 and 64, respectively. The channel is assumed to be frequency selective fading with the number of taps given by $L = 8$.

A. Comparison of Hopping Patterns

We first investigate how hopping patterns affect the resulting BER performance through simulations. Figure 2 compares BER performance of SHMA with random hopping and that with the proposed hopping for the two-user case. In the case of SIC, the result for each of $U_1$ and $U_2$ defined in Section IV-B is plotted. For the random hopping case, the initial subcarrier index is chosen randomly for all OFDM symbols. In the case of the proposed hopping, the initial subcarrier index is chosen randomly for the first OFDM symbol but the remaining indices
are systematically determined as described in Section II-B. The proposed systematic hopping design outperforms the random hopping case due to the increased frequency diversity gain. In fact, it has been shown that the maximum diversity order achieved by SHMA is \( r = \min(L, d_i) \) (i.e., \( r = L \) in our simulated case) [1]. This maximum diversity order may not be necessarily achieved by random hopping, whereas the proposed approach appears to achieve this. In the next subsection, we will verify this by comparing the simulation results with the corresponding theoretical lower bounds.

**B. Comparison between Simulation and Approximate Bound**

Figure 3 compares the simulated BER results and the approximate lower bounds developed in the previous section. The approximate lower bound is calculated by substituting the approximate PEP into the lower bound (12) for each MUD scheme. It can be observed that the approximate lower bounds of STD and \( U_1 \) with SIC show good agreement with the corresponding simulated BER in high SNR. This is because the pairwise error event at the minimum free distance dominates all the error events as SNR increases. This observation also indicates that our proposed hopping design can achieve full diversity gain from the channel. On the other hand, a noticeable gap is observed between the theoretical bound and simulation result for the case of \( U_2 \) with SIC. This stems from the fact that our analysis takes into account only the two tractable pairwise error events at the minimum free distance of \( U_2 \) where the codeword of \( U_1 \) is correctly decoded or \( U_2 \) has errors but with the same pairwise error event as \( U_2 \). Nevertheless, we observe from the simulation results that \( U_1 \) is superior to \( U_2 \) for low SNR but \( U_2 \) eventually outperforms \( U_1 \) as SNR increases, and this tendency is also observed from the comparison of the two theoretical lower bounds, which justifies the comparison of the two theoretical lower bounds.

**VI. Conclusion**

In this paper, we have developed an approximate BER expression for SHMA system employing STD and SIC. We have also proposed a new hopping pattern design that can achieve full diversity gain over static fading channels. Through the numerical results, the validity of our analysis as well as that of the proposed hopping design is demonstrated.

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**REFERENCES**